DUFFING IN VARIABILI ~~"UNIVERSALI"~~ "NORMALI" (1)

$p, q \in \mathbb{R} \quad a \in \mathbb{C}$

$$\ddot{q} + \omega_0^2 q = -\varepsilon \omega_0^2 q^3 \qquad a = \sqrt{\omega_0}\, q + \frac{i}{\sqrt{\omega_0}} p$$

Cerco l'equazione per a

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\omega_0^2 q - \varepsilon \omega_0^2 q^3 \end{cases}$$

$$\dot{a} = \sqrt{\omega_0}\, \dot{q} + \frac{i}{\sqrt{\omega_0}} \dot{p} = \sqrt{\omega_0}\, p + \frac{i}{\sqrt{\omega_0}}\left(-\omega_0^2 q - \varepsilon \omega_0^2 q^3\right) = \sqrt{\omega_0}\, p - i\omega_0 \sqrt{\omega_0}\, q - i\varepsilon \sqrt{\omega_0}\, \omega_0 q^3 =$$

$$= -i\omega_0 \left(\sqrt{\omega_0}\, q + i \frac{1}{\sqrt{\omega_0}} p\right) - i\varepsilon \sqrt{\omega_0}\, \omega_0 q^3 = -i\omega_0 a - i\varepsilon \sqrt{\omega_0}\, \omega_0 q^3$$

$$i\dot{a} = \omega_0 a + \varepsilon \sqrt{\omega_0}\, \omega_0 q^3$$

cerco q in funzione di a

$$\begin{cases} a = \sqrt{\omega_0}\, q + \frac{i}{\sqrt{\omega_0}} p \\ a^* = \sqrt{\omega_0}\, q - \frac{i}{\sqrt{\omega_0}} p \end{cases} \qquad 2\sqrt{\omega_0}\, q = a + a^* \qquad \boxed{q = \frac{a + a^*}{2\sqrt{\omega_0}}}$$

$$i\dot{a} = \omega_0 a + \varepsilon \sqrt{\omega_0}\, \omega_0 \left(\frac{a + a^*}{2\sqrt{\omega_0}}\right)^3 = \omega_0 a + \frac{\varepsilon}{8}\left(a^3 + 3|a|^2 a + 3|a|^2 a^* + a^{*3}\right)$$

$$\boxed{i\dot{a} = \omega_0 a + \frac{3}{8}\varepsilon |a|^2 a + \frac{\varepsilon}{8}\left(a^3 + 3|a|^2 a^* + a^{*3}\right)}$$

con C. i. $a(t=0) = a_0$

Cerchiamo ~~su~~ una soluzione periodica in modo perturbativo (ignorando il problema della ~~se~~ secolarità)

$$a = a^{(0)} + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \dots$$

ε^0 $\quad i\dot{a}^{(0)} = \omega_0 a^{(0)} \quad \Longrightarrow \quad a^{(0)}(t) = a_0 e^{-i\omega_0 t}$

ε^1 $\quad i\dot{a}^{(1)} = \omega_0 a^{(1)} + \frac{3}{8} |a_0|^2 a_0 e^{-i\omega_0 t} + \frac{1}{8}\left(a_0^3 e^{-i3\omega_0 t} + 3|a_0|^2 e^{i\omega_0 t} a_0^* + a_0^{*3} e^{i3\omega_0 t} \right)$

(4) (1) (2) (3)

Soluzione dell'omogenea associata

$$a^{(1)}_{OM} = b_0 e^{-i\omega_0 t} \qquad b_0 \text{ da determinare}$$

Abbiamo ~~3~~ 4 soluzioni particolari, 3 delle quali oscillano con frequenze diverse dalla freq della soluzione dell'omogenea associata

$$a^{(1)}_{P_1} = c_1 e^{-i3\omega_0 t} \qquad a^{(1)}_{P_2} = c_2 e^{i\omega_0 t} \qquad a^{(1)}_{P_3} = c_3 e^{i3\omega_0 t}$$

Per ~~la~~ quella invece in risonanza $\quad a^{(1)}_{P_4} = c_4 t e^{-i\omega_0 t}$

Si scrivono 4 eq.

$$i\dot{a}^{(1)}_{P_1} = \omega_0 a^{(1)}_{P_1} + \frac{1}{8} a_0^3 e^{-i3\omega_0 t} \qquad (1)$$

$$i\dot{a}^{(1)}_{P_2} = \omega_0 a^{(1)}_{P_2} + \frac{3}{8} |a_0|^2 e^{i\omega_0 t} a_0^* \qquad (2)$$

$$i\dot{a}^{(1)}_{P_3} = \omega_0 a^{(1)}_{P_3} + \frac{1}{8} a_0^{*3} e^{i3\omega_0 t} \qquad (3)$$

$$i\dot{a}^{(1)}_{P_4} = \omega_0 a^{(1)}_{P_4} + \frac{3}{8} |a_0|^2 a_0 e^{-i\omega_0 t} \qquad (4)$$

1 $$c_1 3\omega_0 \cancel{e^{-i3\omega_0 t}} = \omega_0 c_1 \cancel{e^{-i3\omega_0 t}} + \frac{1}{8} a_0^3 \cancel{e^{-i3\omega_0 t}}$$

$$c_1 = \frac{a_0^3}{16\omega_0}$$

2 $$-c_2 \omega_0 \cancel{e^{i\omega_0 t}} = \omega_0 c_2 \cancel{e^{i\omega_0 t}} + \frac{3}{8} |a_0|^2 a_0^* \cancel{e^{i\omega_0 t}}$$

$$c_2 = -\frac{3}{16\omega_0} |a_0|^2 a_0^*$$

3 $$-3\omega_0 \cancel{e^{i3\omega_0 t}} c_3 = \omega_0 c_3 \cancel{e^{i3\omega_0 t}} + \frac{1}{8} a_0^{*3} \cancel{e^{i3\omega_0 t}}$$

$$-4\omega_0 c_3 = \frac{1}{8} a_0^{*3} \qquad c_3 = -\frac{1}{32\omega_0} a_0^{*3}$$

4 $$i c_4 \cancel{e^{-i\omega_0 t}} + \cancel{c_4 t e^{-i\omega_0 t} \omega_0} = \cancel{\omega_0 c_4 t e^{-i\omega_0 t}} + \frac{3}{8} |a_0|^2 a_0 e^{-i\omega_0 t}$$

$$c_4 = -i \frac{3}{8} |a_0|^2 a_0$$

quindi

$$\boxed{a^{(1)} = b_0 e^{-i\omega_0 t} - i \frac{3}{8} |a_0|^2 a_0 t e^{-i\omega_0 t} - \frac{3}{16\omega_0} |a_0|^2 a_0^* e^{i\omega_0 t} + \frac{a_0^3}{16\omega_0} e^{-i3\omega_0 t} - \frac{1}{32\omega_0} a_0^{*3} e^{+i3\omega_0 t}}$$

imponendo $a^{(1)}(t=0) = 0$

$$\boxed{b_0 = \frac{3}{16\omega_0} |a_0|^2 a_0^* - \frac{a_0^3}{16\omega_0} + \frac{1}{32\omega_0} a_0^{*3}}$$

TERMINE SECOLARE

SOLUZIONE DIVERGE

Per risolvere il problema della divergenza si considera una frequenza rinormalizzata

$$i\dot{a} = \Omega a + \frac{\varepsilon}{8}\left(a^3 + 3|a|^2 a^* + a^{*3}\right) \qquad \text{con } \Omega = \omega_0 + \varepsilon\frac{3}{8}|a|^2$$

$$a = a^{(0)} + \varepsilon a^{(1)}$$

ε^0

$$i\dot{a}^{(0)} = \Omega a^{(0)} \qquad a^{(0)} = a_0 e^{-i\Omega_0 t} = a_0 e^{-i(\omega_0 + \frac{3}{8}|a_0|^2)t} \qquad \Omega_0 = \omega_0 + \varepsilon\frac{3}{8}|a_0|^2$$

ε^1

$$i\dot{a}^{(1)} = \Omega_0 a^{(1)} + \frac{1}{8}\left(\underset{P_1}{a_0^3 e^{-i3\Omega_0 t}} + \underset{P_2}{3|a_0|^2 a_0^* e^{i\Omega_0 t}} + \underset{P_3}{a_0^{*3} e^{i3\Omega_0 t}}\right)$$

$$a^{(1)} = a^{(1)}_{OM} + a^{(1)}_{P_1} + a^{(1)}_{P_2} + a^{(1)}_{P_3}$$

$$a^{(1)}_{OM} = b_0 e^{-i\Omega_0 t}$$

- $a^{(1)}_{P_1} = C_1 e^{-i3\Omega_0 t} \qquad 3\Omega_0 C_1 = \Omega_0 C_1 + \frac{1}{8}a_0^3 \qquad C_1 = \frac{1}{16\Omega_0}a_0^3 \Rightarrow a^{(1)}_{P_1} = \frac{1}{16}\frac{a_0^3}{\Omega_0}e^{-i3\Omega_0 t}$
- $a^{(1)}_{P_2} = C_2 e^{i\Omega_0 t} \qquad -\Omega_0 C_2 = \Omega_0 C_2 + \frac{3}{8}|a_0|^2 a_0^* \qquad C_2 = -\frac{3|a_0|^2 a_0^*}{16\Omega_0} \Rightarrow a^{(1)}_{P_2} = -\frac{3|a_0|^2 a_0^*}{16\Omega_0}e^{i\Omega_0 t}$
- $a^{(1)}_{P_3} = C_3 e^{i3\Omega_0 t} \qquad -3\Omega_0 C_3 = \Omega_0 C_3 + \frac{1}{8}a_0^{*3} \qquad C_3 = -\frac{1}{32\Omega_0}a_0^{*3} \Rightarrow a^{(1)}_{P_3} = -\frac{1}{32}\frac{a_0^{*3}}{\Omega_0}e^{i3\Omega_0 t}$

La soluzione dunque è

$$a = a_0 e^{-i\Omega_0 t} + \varepsilon\left(b_0 e^{-i\Omega_0 t} + \frac{1}{16}\frac{a_0^3}{\Omega_0} e^{-i3\Omega_0 t} - \frac{3}{16}\frac{|a_0|^2 a_0^*}{\Omega_0} e^{i\Omega_0 t} - \frac{1}{32}\frac{a_0^{*3}}{\Omega_0} e^{i3\Omega_0 t}\right) + \cancel{\varepsilon^2}\cdots$$

con $\quad b_0 = \dfrac{3}{16}\dfrac{|a_0|^2 a_0^*}{\Omega_0} + \dfrac{1}{32}\dfrac{a_0^{*3}}{\Omega_0} - \dfrac{1}{16}\dfrac{a_0^3}{\Omega_0}$

Tornando alla variabile $q = \dfrac{a + a^*}{2\sqrt{\omega_0}}$ si dovrebbe tornare alla soluzioni ottenuta precedentemente.